Erratum: Self-Averaging and Ergodicity of Anomalous Diffusion in Quenched Random Media [Phys. Rev. E 93, 010101(R) (2016)]

Marco Dentz
Spanish National Research Council (IDAEA-CSIC), 08034 Barcelona, Spain[*]

Anna Russian and Philippe Gouze
Géosciences, Université de Montpellier 2, CNRS, Montpellier, France
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In our paper we studied self-averaging and ergodicity for anomalous diffusion in quenched random media. We concluded that diffusion is both self-averaging and ergodic in $d \geq 2$ and non-self-averaging and non-ergodic in $d < 2$. While our main results regarding the self-averaging property remain unchanged, we revise here the statement on ergodicity, correct the calculation that led to it, and develop the correct results for the ergodicity property. The time-average mean square displacement is in fact weakly non-ergodic, which is consistent with Refs. [1–4]. In order to clarify these points, we briefly restate the problem setup, and the definitions of the noise average and time-average mean square displacements.

Particle motion is described by the Langevin equation

$$d\mathbf{x}(t) = \sqrt{\frac{2\kappa dt}{\theta[\mathbf{x}(t)]}}\boldsymbol{\xi}(t), \tag{1}$$

where $\mathbf{x}(t)$ is the position of a diffusing particle, the $\boldsymbol{\xi}(t)$ are identical independently distributed Gaussian random variables with a mean of 0 and unit variance, κ is the constant diffusion coefficient. The mobility $\theta(\mathbf{x})$ represents the quenched disorder. This model is equivalent to a quenched random trap model, which can be seen, by performing the transformation $ds = \theta[\mathbf{x}(t)]^{-1}dt$ such that

$$d\mathbf{x}(s) = \sqrt{2\kappa ds}\boldsymbol{\xi}(s), \qquad dt(s) = \theta[\mathbf{x}(s)]ds. \tag{2}$$

Equation (2) is coarse-grained on the characteristic length scale ℓ, which gives the recursion relation

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \ell\boldsymbol{\eta}_n, \qquad t_{n+1} = t_n + \theta(\mathbf{x}_n)\hat{\tau}_n, \tag{3}$$

as given by Eq. (7) in the manuscript; the $\hat{\tau}_n$ are identical independently distributed exponential random variables with the characteristic time $\tau_\kappa = \ell^2/(2\kappa)$. The particle position is now given by $\mathbf{x}(t) = \mathbf{x}_{n_t}$ with $n_t = \sup(n|t_n \leq t)$ the number of steps needed to reach time t. The operational time $s(t) = s_{n_t}$, where $s_n = \sum_{i=0}^{n-1}\hat{\tau}_i$ is Gamma-distributed with mean $n\tau_\kappa$. Thus in the following, we set $s(t) = n_t\tau_\kappa$.

The mean square displacement in a single medium realization it given by the noise average $m(t) = \langle \mathbf{x}(t)^2 \rangle$. We obtain from (1) by using the Ito interpretation,

$$m(t) = 2\kappa d\left\langle \int\limits_0^t dt'\theta[\mathbf{x}(t')]^{-1} \right\rangle = 2\kappa d\langle s(t)\rangle = d\ell^2\langle n_t\rangle, \tag{4}$$

where we used $ds = \theta[\mathbf{x}(t)]^{-1}dt$. The time averaged mean square displacement is defined as

$$m_\Delta(t) = \frac{1}{t-\Delta}\int\limits_0^{t-\Delta} dt'\mathbf{x}_\Delta(t)^2 \tag{5}$$

where $\mathbf{x}_\Delta(t)$ is given by

$$\mathbf{x}_\Delta(t) = \int\limits_t^{t+\Delta} dt'\sqrt{2\kappa\theta[\mathbf{x}(t')]^{-1}}\boldsymbol{\zeta}(t') \tag{6}$$

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* marco.dentz@csic.es

with $\boldsymbol{\zeta}(t)$ a Gaussian white noise. We consider the limit of $\Delta/t \ll 1$, for which we obtain

$$\mathbf{x}_\Delta(t) = \sqrt{2\kappa\theta[\mathbf{x}(t)]^{-1}}\boldsymbol{w}_\Delta(t), \qquad \boldsymbol{w}_\Delta(t) = \int_t^{t+\Delta} dt' \boldsymbol{\zeta}(t'), \tag{7}$$

with $\langle \boldsymbol{w}_\Delta(t) \rangle = \mathbf{0}$ and $\langle \boldsymbol{w}_\Delta(t) \cdot \boldsymbol{w}_\Delta(t') \rangle = d\Delta$ if $|t - t'| < \Delta$ and 0 else. Thus, we obtain for $m_\Delta(t)$

$$m_\Delta(t) = \frac{2\kappa d\Delta}{t} \int_0^t dt' \theta[\mathbf{x}(t')]^{-1} = \frac{2\kappa d\Delta}{t} s(t) = \frac{d\ell^2\Delta}{t} n_t, \tag{8}$$

where we set $w_\Delta(t)^2 = d\Delta$. In order to study the ergodicity of the diffusion process, we consider the variance of $m_\Delta(t)$ with respect to its noise average

$$\sigma^2_\Delta(t) = \langle m_\Delta(t)^2 \rangle - \langle m_\Delta(t) \rangle^2, \tag{9}$$

where $\langle m_\Delta(t) \rangle$ is given by

$$\langle m_\Delta(t) \rangle = \frac{m(t)\Delta}{t}. \tag{10}$$

The noise mean square of $m_\Delta(t)$ is given by

$$\langle m_\Delta(t)^2 \rangle = \frac{4\kappa^2 d^2 \Delta^2}{t^2} \left\langle \int_0^t dt' \theta[\mathbf{x}(t')]^{-1} \int_0^t dt'' \theta[\mathbf{x}(t'')]^{-1} \right\rangle = \frac{4\kappa^2 d^2\Delta^2}{t^2} \left\langle s(t)^2 \right\rangle = \frac{d^2\ell^4\Delta^2}{t^2} \left\langle n_t^2 \right\rangle. \tag{11}$$

This equation corrects Equation (13) in the Supplementary Material of our paper, in which we erroneously stated that $\langle m_\Delta(t)^2 \rangle = m(t)^2\Delta^2/t^2$. This, however disregards correlations between $\theta[\mathbf{x}(t')]$ and $\theta[\mathbf{x}(t'')]$. Based on this statement, we concluded that the noise variance $\sigma^2_\Delta(t)$ of the time average mean square displacement in a single disorder realization was 0 and thus that diffusion was ergodic. We clarify this in the following.

In our paper, we analyze both numerically and analytically the variance $\sigma^2_m(t) = \overline{m(t)}^2 - \overline{m}(t)^2$ and the relative variance

$$\Sigma(t) = \frac{\sigma^2_m(t)}{\overline{m}(t)^2} \tag{12}$$

which probes the disorder sample to sample fluctuations of $m(t)$, the noise average mean square displacement. We find in our paper that $\Sigma(t)$ goes asymptotically to 0 for $d \geq 2$, which means that $m(t)$ is self-averaging. For $d < 2$, $\Sigma(t)$ goes towards a constant, which means that $m(t)$ is not self-averaging. While these statements are true for the self-averaging property of $m(t)$, they are not for the ergodicity of $m_\Delta(t)$, the time average mean square displacement.

In order to probe ergodicity, we quantify the noise variance (9). Notice that the noise variance $\sigma^2_\Delta(t)$ fluctuates between disorder realizations. However, we have shown in our paper that the ensemble average is asymptotically a good estimator for the noise average, at least for the mean square displacement $m(t)$ in $d \geq 2$ dimensions because $m(t)$ is self-averaging. Based on this, we use the ensemble average $\overline{\sigma^2_\Delta(t)}$ as an estimator for $\sigma^2_\Delta(t)$ and the ensemble average $\overline{\langle m_\Delta(t) \rangle}$ as an estimator for $\langle m_\Delta(t) \rangle$ in $d \geq 2$. Using (8) and (10) in (9) and performing the disorder average, we obtain

$$\overline{\sigma^2_\Delta(t)} = \frac{\Delta^2}{t^2}\left[d^2\ell^4\overline{\langle n_t^2 \rangle} - \overline{m(t)^2}\right]. \tag{13}$$

We rewrite the latter in the form

$$\overline{\sigma^2_\Delta(t)} = \frac{d^2\ell^4\Delta^2}{t^2}\left[\overline{\langle n_t^2 \rangle} - \overline{\langle n_t \rangle}^2\right] - \frac{\Delta^2}{t^2}\left[\overline{m(t)^2} - \overline{m(t)}^2\right], \tag{14}$$

where we note that $\overline{m(t)}^2 = d^2\ell^4\overline{\langle n_t \rangle}^2$. Notice that the first expression in square brackets denotes the disorder variance $\sigma^2_n(t)$ of the number of steps n_t to reach time t. The second term in square brackets denotes the disorder variance $\sigma^2_m(t)$ of the noise average mean square displacement $m(t)$. Thus, we can restate (14) as

$$\overline{\sigma^2_\Delta(t)} = \frac{\Delta^2}{t^2}\left[d^2\ell^4\sigma^2_n(t) - \sigma^2_m(t)\right]. \tag{15}$$

This relation implies that at finite times $\sigma_n^2(t) \geq \sigma_m^2(t)/(d^2\ell^4) > 0$. Thus, in order to study the ergodicity property for $d \geq 2$, we now focus on the variance $\sigma_n^2(t)$ of n_t [1–3], the number of steps to reach time t.

We follow the methodology developed in our paper in order to determine explicit results for $\sigma_n^2(t)$. The disorder ensemble expectation $\overline{n_t}$ is encoded in $\overline{m}(t)$, see Equation (31) in the Supplementary Material of our paper. The disorder ensemble expectation of $\overline{n_t^2}$ is given by

$$\overline{n_t^2} = \sum_{n=0}^{\infty} n^2 \overline{\delta_{n,n_t}} = \sum_{n=0}^{\infty} n^2 \overline{\mathbb{I}(t_n \leq t < t_{n+1})}. \tag{16}$$

It is independent on the noise such that we can omit the noise averages in the variance $\sigma_n^2(t)$. We evaluate the scaling of this sum by using expression (29) developed in the Supplementary Material of our paper for the average of the indicator function, which reads as

$$I_n(t) \equiv \overline{\mathbb{I}(t_n \leq t < t_{n+1})} = \frac{t}{\alpha_n} \frac{d \ln(\alpha_n)}{dn} f_\beta(t/\alpha_n), \tag{17}$$

where $\alpha_n = n S_n^{\frac{1-\beta}{\beta}}$; S_n is the average number of distinct sites visited by a random walker, which depends on the dimension of space. For $d = 2$, we find that

$$\overline{n_t^2} \propto t^{2\beta} \ln(t)^{2-2\beta} \propto \overline{n_t}^2. \tag{18}$$

This implies that $\sigma_n^2(t) \propto t^{2\beta} \ln(t)^{2-2\beta}$ because at finite times $\sigma_n^2(t) > 0$. And for $d > 2$, we obtain

$$\overline{n_t^2} \propto t^{2\beta} \propto \overline{n_t}^2, \tag{19}$$

which implies that $\sigma_n^2(t) \propto t^{2\beta}$. Thus, $\sigma_n^2(t) > \sigma_m^2(t)/(d^2\ell^4)$, compare to Eqs. (21) and (22) in our paper. This implies that

$$\overline{\sigma_\Delta^2(t)} = \frac{\Delta^2}{t^2} d^2 \ell^4 \sigma_n^2(t) + \dots. \tag{20}$$

Furthermore, the disorder average of $\langle m_\Delta(t) \rangle$ is given by

$$\overline{\langle m_\Delta(t) \rangle} = \frac{\Delta}{t} \overline{m}(t) = \frac{\Delta}{t} d\ell^2 \overline{n_t}. \tag{21}$$

This implies that the ergodicity breaking parameter of Ref. [2], which in our notation reads as

$$\mathrm{EB} = \frac{\overline{\sigma_\Delta^2(t)}}{\overline{\langle m_\Delta(t) \rangle}^2} = \frac{\sigma_n^2(t)}{\overline{n_t}^2} + \dots \tag{22}$$

goes towards a constant for $t \to \infty$. This implies that the time-average mean square displacement is weakly non-ergodic in $d \geq 2$, which is consistent with Refs. [1–4]. For $d < 2$, we cannot make a statement on the ergodicity based on the disorder averages of $\sigma_\Delta^2(t)$ and $\langle m_\Delta(t) \rangle$ because $m(t)$ is not self-averaging.

In summary, unlike stated in our paper, the time-average mean square displacement $m_\Delta(t)$ is weakly non-ergodic for $d \geq 2$. The noise mean square displacement $m(t)$, on the other hand is self-averaging for $d \geq 2$ and non-self-averaging in $d < 2$, as found in our paper.

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[1] G. Bel and E. Barkai, Phys. Rev. Lett. **94**, 240602 (2005).
[2] Y. He, S. Burov, R. Metzler, and E. Barkai, Phys. Rev. Lett. **101**, 058101 (2008).
[3] T. Miyaguchi and T. Akimoto, Phys. Rev. E **83**, 031926 (2011).
[4] P. Massignan, C. Manzo, J. A. Torreno-Pina, M. F. García-Parajo, M. Lewenstein, and J. Lapeyre, G. J., Phys. Rev. Lett. **112**, 150603 (2014).